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Reliability-Based Optimization of Uncertain Systems in Structural Dynamics

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The reliability-based optimization of uncertain linear structural systems subjected to stochastic excitation is considered. Uncertain system parameters are modeled as random variables with prescribed joint probability density function. Second-order probabilistic descriptors are combined with approximate extreme response theories to obtain conditional reliability estimates for the system. Approximations based on asymptotic expansions are used to provide a computationally efficient estimate for the unconditional system reliability that accounts for the uncertainties in the system parameters. A general solution strategy for the corresponding reliability-based optimization problem is presented. Implementation issues related to the evaluation of system response functions and calculation of design points are addressed. The effects of uncertainty in the system parameters, as well as the accuracy of reliability estimates on the optimal design, are investigated. It is shown that these two factors are important because they can change the optimal design significantly. A generic primary–secondary system is presented to illustrate the performance and efficiency of the proposed implementation.

Nomenclature

$\{b\}$	=	vector of uncertain system parameters
$C(\cdot)$	=	total cost function
$[C]$	=	damping matrix
c_a, c_p	=	damping coefficients of the absorber and primary systems
$E_t(\cdot)$	=	expectation operation in time domain
$[G]$	=	compatibility matrix
$[H(\cdot)]$	=	Hessian matrix
$J(\cdot)$	=	unconditional quantity
$[K]$	=	stiffness matrix
k_a, k_p	=	stiffnesses of the absorber and primary systems
$[M]$	=	mass matrix
m_a, m_p	=	masses of the absorber and primary systems
$\{n(\cdot)\}$	=	Gaussian white noise excitation vector
$P(\cdot)$	=	probability density function
$P_F(\cdot)$, $P_R(\cdot)$	=	failure probability and reliability function
$\{q(\cdot)\}$	=	state-space vector
$r_i(\cdot)$	=	system response
S_i, T_i, U_i	=	modal energies
$s_i(\cdot)$	=	modulating time function
t	=	time variable
$\{u(\cdot)\}, \{\dot{u}(\cdot)\}, \{\ddot{u}(\cdot)\}$	=	displacement, velocity, and accelaration response vectors
$\{x\}$	=	vector of design variables
$\Gamma_{ij}(\cdot)$	=	modal cross covariances
$\eta_i(\cdot)$	=	modal participation coefficient
κ_i	=	threshold level
λ_i	=	eigenvalue
$\nu^+(\cdot)$	=	expected rate of up-crossing a threshold level
$\sigma^2_{r_i}, \sigma^2_{\dot{r}_i}, \sigma_{r_i\dot{r}_i}$	=	second-order statistics
$\Phi(\cdot)$	=	Gaussian distribution function
$\{\chi\}_i, \{\phi\}_i$	=	left and right eigenvectors
$\{\chi\}_{pi}, \{\phi\}_{pi}$	=	position parts of the left and right eigenvectors

Introduction

OPTIMIZATION via general nonlinear mathematical programming techniques has been widely accepted as a viable methodology for engineering design. It is clear that, when a structure is being designed, the environmental loads that the built structure will experience in its lifetime are highly uncertain. The uncertain load time history needed in the dynamic analysis of a structure subjected to environmental loads such as aerodynamic turbulence, wind, water wave excitation, and earthquake is an uncertain value function, and it is best modeled by a stochastic process.[1,2] Likewise, response predictions are made during design based on structural models, whose parameters are uncertain because the properties that will be exhibited by the structure when completed are not known precisely. These uncertainties result from the numerous assumptions made when modeling the geometry, the boundary conditions, constitutive behavior of the materials involved, etc. Probabilistic methods provide the means for incorporating system uncertainties in the analysis by describing the uncertainties as random variables with a prescribed joint probability density function. Uncertainties in both loading and structural characteristics can adversely affect the reliability of the structure. Therefore, it is necessary to consider their effects explicitly during the optimization process to achieve a balance between cost and safety for the optimal design.[3,4]

In reliability-based structural optimization, the total expected cost related to the structure, including the initial, maintenance, and failure costs, is usually used as an objective function. The constraints are reliability requirements with respect to the possible failure modes of the structure. If the structural characteristics are known, the conditional reliability estimates can be calculated using well-known techniques from random vibration theory. System reliabilities that account for the uncertainties in the system parameters are given by the total probability theorem as particular multidimensional integrals over the space of uncertain parameters. Exact analytical solutions for these unconditional multidimensional integrals can only be found for a very limited number of simple systems. For more realistic systems, simulation techniques such as Monte Carlo and importance sampling can be used to provide accurate results for evaluating unconditional system reliabilities.[5] Other methods that have been developed to provide computational tools for approximating reliabilities of uncertain systems subjected to stochastic loads are the first-order reliability method[6] and second-order reliability method (SORM).[7] These methods have been tested for a variety of structural problems, including simple linear and nonlinear systems and primary–secondary systems. Additional methods that have been developed for computing unconditional system reliabilities include the perturbation method[8] and the asymptotic method.[9,10] The perturbation method is the least expensive method from a computational point of view, but it works well only for a limited number of cases. The asymptotic method, on the other hand, is a technique based on the Laplace's method for asymptotic approximation of integrals.

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One important feature of this technique is the use of simple analytical formulas for reliability estimates, which makes the method simpler than other existing SORM.

Reliability-based optimization problems can be characterized as two-level optimization problems. Level 1 is the overall optimization in the design variables, and level 2 is the failure and reliability estimates. For realistic systems, these estimates completely dominate the total calculation cost. Therefore, the number of system response calculations, including reliability estimates, should be as small as possible. In this context, one promising technique is the asymptotic approximation method because of its simplicity and expediency. In this case, the sublevel optimization problem becomes an unconstrained optimization problem.[11] Approximation concepts[12,13] are used to develop an efficient numerical implementation for the solution of the sublevel optimization problem. In the approximation concepts method, a sequence of approximate optimization problems is generated, and they are solved using conventional optimization techniques.

First, the formulation of the reliability-based optimization problem is presented. Next, reliability estimates in terms of the asymptotic method are reviewed. Solution strategies for an efficient numerical implementation of the methodology are then discussed. Finally, a test problem is considered to illustrate the ideas set forth.

Problem Formulation

Let the vectors $\{x\}$ $(x_i, i = 1, \ldots, n)$ and $\{b\}$ $(b_i, i = 1, \ldots, m)$ represent the vector of design variables and uncertain system parameters, respectively. The uncertain system parameters are modeled using a prescribed joint probability density function $P(\{b\})$. This function indicates the relative plausibility of the possible values of the uncertain parameters $\{b\} \in \Omega$, with Ω being a subset of R^m. In reliability-based structural optimization, there will usually be code specifications with requirements for the reliability of the structural components and/or the total system. In this formulation, the constraints are related to single failure modes. Then, the structural synthesis problem can be written as a two-level nonlinear mathematical programming problem of the form

$$\min_{\{x\}} C(\{x\})$$

subject to the design constraints

$$P_{F_i}(\{x\}) \le P^*_{F_i}, \qquad i = 1, \ldots, K$$
$$G_j(\{x\}) \le 0, \qquad j = 1, \ldots, M, \qquad \{x\} \in \Delta \tag{1}$$

where $C(\{\cdot\})$ is the total cost function, including initial construction costs and expected failure costs, $P_{F_i}(\{\cdot\})$ is the failure probability function for failure mode number i, $P^*_{F_i}$ is a user specified level of failure, G_j is a deterministic constraint related to general design requirements, and Δ is the set that contains the side constraints for the vector of design variables $\{x\}$. The top-level optimization problem is the overall optimization in the design variables, whereas the sublevel problem corresponds to the failure probability estimates. Note that the two levels separate, although the two types of variables are nested in the problem. For example, when failure probability calculations are performed, it is for fixed values $x_i, i = 1, \ldots, n$. The total cost function $C(\{x\})$ and the failure probability functions $P_{F_i}(\{x\}), i = 1, \ldots, K$, represent unconditional quantities for the design $\{x\}$. That is, they account for the uncertainties in the system parameters as well as the uncertainties in the loads. These quantities can be written in terms of conditional quantities by using the total probability theorem as

$$C(\{x\}) = \int_\Omega C(\{x\} \mid \{b\}) P(\{b\})\, \mathrm{d}\{b\}$$
$$P_{F_i}(\{x\}) = \int_\Omega P_{F_i}(\{x\} \mid \{b\}) P(\{b\})\, \mathrm{d}\{b\}, \qquad i = 1, \ldots, K \tag{2}$$

where $C(\{x\} \mid \{b\})$ is the conditional total cost and $P_{F_i}(\{x\} \mid \{b\})$, $i = 1, \ldots, K$ are the conditional failure probabilities for the design $\{x\}$, given the vector of system parameters $\{b\}$.

Approximations for Reliability Estimates

The unconditional objective function as well as the unconditional constraint functions defined in Eq. (2) are multidimensional probability integrals that rarely, if ever, can be integrated analytically. In the proposed implementation, asymptotic approximations of multidimensional integrals are used to derive estimates for the unconditional quantities. For the sake of completeness, the basic ideas of this technique are reviewed briefly.[9,10] The asymptotic approximation is based on the expansion of the logarithm of the integrand of the conditional quantities about the point that corresponds to the maximum of the integrand. The value of $\{b\}$ that maximizes the integrand $\{b^*\}$ is called a design point. Let $J(\{x\})$ denote an unconditional quantity of the form

$$J(\{x\}) = \int_\Omega J(\{x\} \mid \{b\}) P(\{b\})\, \mathrm{d}\{b\} \tag{3}$$

where $J(\{x\} \mid \{b\})$ is a conditional quantity for the design $\{x\}$ given the vector of system parameters $\{b\}$. It is clear that the total cost function and the failure probability functions of problem (1) have the representation given in Eq. (3). When a second-order expansion of $\ln(J(\{x\} \mid \{b\}) P(\{b\}))$ about $\{b^*\}$ is considered, and when it is noted that its derivatives are zero at $\{b^*\}$, it is found that

$$\begin{aligned} J(\{x\}) &= \int_\Omega \exp[\ln(J(\{x\} \mid \{b\}) P(\{b\}))]\, \mathrm{d}\{b\} \\ &= J(\{x\} \mid \{b^*\}) P(\{b^*\}) \\ &\times \int_\Omega \exp\left[-\frac{1}{2}(\{b\} - \{b^*\})^t [H(\{b^*\})](\{b\} - \{b^*\})\right] \\ &\times \exp[R(\{b\})]\, \mathrm{d}\{b\} \end{aligned} \tag{4}$$

where $[H(\{b^*\})]$ is the Hessian matrix of $-\ln[J(\{x\} \mid \{b\}) P(\{b\})]$ evaluated at the design point $\{b^*\}$ and $R(\{b\})$ is the expansion error. In Eq. (4) it is assumed that $\{b^*\}$ occurs inside the region Ω. Finally, applying Laplace's method of asymptotic expansion to the integral in Eq. (4) and noting that $R(\{b^*\}) = 0$ gives an asymptotic approximation for $J(\{x\})$ as[14]

$$J(\{x\}) \approx (2\pi)^{m/2} \frac{J(\{x\} \mid \{b^*\}) P(\{b^*\})}{\sqrt{|[H(\{b^*\})]|}} \tag{5}$$

where $|[H(\{b^*\})]|$ is the determinant of the Hessian matrix. The approximation given in Eq. (5) is asymptotically correct, that is, the sharper the peak of the integrand is about its maximum value, the more accurate the value of the approximation is expected to be. In fact, it can be shown that the error in the approximation converges to zero as the smallest eigenvalue of $[H(\{b^*\})]$ tends to infinity.[15] In the case of multiple design points $\{b^*\}_i$, $i = 1, \ldots, L$, the asymptotic approximation is given by summing the contribution for each design point, that is,

$$J(\{x\}) = \sum_{i=1}^{L} J_i(\{x\})$$

where $J_i(\{x\}), i = 1, \ldots, L$, is the asymptotic contribution to the unconditional quantity from the design point $\{b^*\}_i$ and is given by

$$J_i(\{x\}) \approx (2\pi)^{m/2} \frac{J(\{x\} \mid \{b^*\}_i) P(\{b^*\}_i)}{\sqrt{|[H(\{b^*\}_i)]|}} \tag{6}$$

Based on the asymptotic results given, Eq. (6) is taken as an approximation for the unconditional quantity $J(\{x\})$. Numerical results have shown that, in general, the asymptotic method gives acceptable and reasonable quantitative results for the type of probability integrals encountered in this formulation.[10] It is emphasized that the unconditional quantity $J(\{x\})$ can be estimated by any other available technique. For example, simulation techniques can improve the value of the multidimensional integrals to any desirable degree of accuracy at the expenses of more computation effort.

Application

In this study, attention is directed toward problems in which the stochastic excitation is a Gaussian white noise process with zero mean. Because of its mathematical simplicity, this type of stochastic process is often used as an approximation to a great number of physical phenomena. The probability that design conditions are satisfied during projected lifetime T provides a useful measure of system performance or reliability. In problem (1), the reliability constraints are related to single failure modes, where failure mode number i is assumed to occur when a system response $r_i(t, \{x\} \mid \{b\})$ reaches some critical level κ_i for the first time. In this context, $r_i(t, \{x\} \mid \{b\})$ is a conditional response quantity for the design $\{x\}$ given the vector of system parameters $\{b\}$. The probability that $r_i(t, \{x\} \mid \{b\})$ has not reached the level κ_i before time T can be obtained using available results from random vibration theory.[1] These results are based on the expected rate of up-crossing and down-crossing through levels κ_i and $-\kappa_i$, respectively. The expected rate of up-crossing a given level κ_j, $\nu^+(t, \{x\} \mid \{b\})$, is given in terms of the second-order statistics $\sigma_{r_i}^2 = E_t[r_i^2(t, \{x\} \mid \{b\})]$, $\sigma_{\dot{r}_i}^2 = E_t[\dot{r}_i^2(t, \{x\} \mid \{b\})]$, and $\sigma_{r_i\dot{r}_i} = E_t[r_i(t, \{x\} \mid \{b\})\dot{r}_i(t, \{x\} \mid \{b\})]$ as[2]

$$\nu^+(t, \{x\} \mid \{b\}) = \frac{\sigma_{\dot{r}_i}(1-s^2)^{\frac{1}{2}}}{2\pi\sigma_{r_i}} \exp\left[\frac{-\kappa_i^2}{2\sigma_{r_i}^2(1-s^2)}\right] + \frac{s\kappa_i\sigma_{\dot{r}_i}}{(2\pi)^{\frac{1}{2}}\sigma_{r_i}^2} \exp\left[\frac{-\kappa_i^2}{2\sigma_{r_i}^2}\right] \Phi\left[\frac{s\kappa_i}{\sigma_{r_i}(1-s^2)^{\frac{1}{2}}}\right] \tag{7}$$

where $E_t(\cdot)$ is the mathematical expectation with respect to uncertainty in the time domain; $s = \sigma_{r_i\dot{r}_i}/\sigma_{r_i}\sigma_{\dot{r}_i}$ is the coefficient of correlation between the response $r_i(t, \{x\} \mid \{b\})$ and its time derivative $\dot{r}_i(t, \{x\} \mid \{b\})$; $\Phi(\cdot)$ is the Gaussian distribution function; σ_{r_i} and $\sigma_{\dot{r}_i}$ are the standard deviation of the response $r_i(t, \{x\} \mid \{b\})$ and its time derivative, respectively; and $\sigma_{r_i\dot{r}_i}$ is the cross correlation between $r_i(t, \{x\} \mid \{b\})$ and $\dot{r}_i(t, \{x\} \mid \{b\})$. For a high threshold level κ_i, it can be assumed that the events of crossing such a level occur independently according to a Poisson process with mean rate $\nu^+(t, \{x\} \mid \{b\})$, in which case the conditional failure probability can be approximated by[1]

$$P_{F_i}(\{x\} \mid \{b\}) = 1 - \text{prob}\left[\max_{[0,T]} |r_i(t, \{x\} \mid \{b\})| \leq \kappa_i\right] \approx 1 - \exp\left(-2\int_0^T \nu^+(\tau, \{x\} \mid \{b\})\, d\tau\right) \tag{8}$$

where prob[·] denotes the probability that the expression in parenthesis is true.

Modal-Based Solution

The solution of the nonlinear mathematical programming problem defined in Eq. (1) requires the evaluation of conditional and unconditional quantities. The conditional quantities $C(\{x\} \mid \{b\})$ and $P_{F_i}(\{x\} \mid \{b\})$, $i = 1, \ldots, K$, require the evaluation of second-order statistics of system response functions. These second-order statistics can be written in terms of the solution of a general underdamped linear system.[16] The derivation of the basic equations are repeated here for the continuity of the formulation. The equation of motion of an l-degree-of-freedom linear structural system subjected to external forces can be cast in the form

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + [K]\{u(t)\} = [G]\{n(t)\} \tag{9}$$

where $\{u(t)\}$, $\{\dot{u}(t)\}$, and $\{\ddot{u}(t)\}$ are the displacement, velocity, and acceleration response vectors of dimension l, respectively; $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices of dimension $l \times l$; $[G]$ is a matrix of dimension $l \times l_G$ relating the force to the degrees of freedom of the system; and $\{n(t)\}$ is a zero-mean Gaussian white noise excitation vector of dimension l_G. In general, the matrices $[M]$, $[C]$, $[K]$, and $[G]$ depend on the vector of design variables $\{x\}$ and uncertain system parameters $\{b\}$. Therefore, the system responses are also functions of $\{x\}$ and $\{b\}$. The solution of Eq. (9) is carried out by standard modal analysis. The equation is recast into the first-order $2l$ state-space form, and the solution is represented as a linear combination of complex mode shapes of the form

$$\{q(t)\} = \sum_{i=1}^{2l} \{\phi\}_i \eta_i(t)$$

where $\{q(t)\}^t = \langle\{\dot{u}(t)\}^t\{u(t)\}^t\rangle$ is the state vector; $\eta_i(t)$, $i = 1, \ldots, 2l$ are the modal participation coefficients; and $\{\phi_i\}$, $i = 1, \ldots, 2l$ are the complex right eigenvectors corresponding to the $2l$ state-space equation. The eigenvalues of the $2l$ state-space equation of motion, λ_i, $i = 1, \ldots, 2l$, can be written as[17]

$$\lambda_i = \frac{-S_i \pm \sqrt{S_i^2 - 4T_iU_i}}{2T_i} \tag{10}$$

where T_i, S_i, and U_i are the modal energies defined as $T_i = \{\chi\}_{pi}^t[M]\{\phi\}_{pi}$, $U_i = \{\chi\}_{pi}^t[K]\{\phi\}_{pi}$, and $S_i = \{\chi\}_{pi}^t[C]\{\phi\}_{pi}$ and where $\{\phi\}_{pi}$ and $\{\chi\}_{pi}$ are the position parts (the last l components) of the right and left eigenvectors of the $2l$ state-space equation of motion, respectively. For underdamped systems, the modes appear in complex conjugate pairs, and so the modal participation coefficients can be arranged to appear in complex conjugate pairs. These coefficients satisfy a first-order differential equation that can be written in terms of the modal energies as[1,17]

$$\dot{\eta}_i(t) - \lambda_i\eta_i(t) = \frac{\{\chi\}_{pi}^t[G]\{n(t)\}}{(2\lambda_iT_i + S_i)}, \qquad i = 1, \ldots, 2l \tag{11}$$

Response Statistics

Denote by $r(t, \{x\} \mid \{b\}) = \{\beta(\{x\} \mid \{b\})\}^t\{u(t)\}$ a conditional response quantity of interest given as a linear combination of the components of the displacement response vector. The dependence of the vector $\{\beta(\cdot)\}$ on the vector of design variables and uncertain system parameters arises when, for example, the response quantity is a stress or force member and some of the cross-sectional properties (dimensions or mechanical properties) of the member are design variables or uncertain system parameters. The state vector $\{q(t)\}^t = \langle\{\dot{u}(t)\}^t\{u(t)\}^t\rangle$ is a zero-mean Gaussian process due to the linearity of the system, and it is fully described by its covariance matrix. The second-order statistics of the response can be written in terms of modal cross covariances. For example, the variance of the response process $r(t, \{x\} \mid \{b\})$ is given by

$$\sigma_r^2 = \sum_{i=1}^{2l}\sum_{j=1}^{2l} \Gamma_{ij}(t)$$

where the quantities $\Gamma_{ij}(t) = E_t[\gamma_i(t)\gamma_j(t)]$ are modal cross covariances, with $\gamma_i(t) = \{\beta\}^t\{\phi\}_{pi}\eta_i(t)$. To derive the equation for the modal cross covariances $\Gamma_{ij}(t)$, it is assumed, without loss of generality, that the force components $n_i(t)$, $i = 1, \ldots, l_G$, of the vector $\{n(t)\}$ are independent with autocorrelation functions $E_t[n_i(t)n_i(\tau)] = \delta(t-\tau)s_i(t)s_i(\tau)$, $i = 1, \ldots, l_G$, where $\delta(\cdot)$ is the delta function and $s_i(\cdot)$ is a deterministic modulating time function. In this case, the cross-covariance function $\Gamma_{ij}(t)$ satisfies the Lyapunov equation that can be written as[1,16]

$$\frac{d}{dt}\Gamma_{ij}(t) - (\lambda_i + \lambda_j)\Gamma_{ij}(t) = \frac{\{\beta\}^t\{\phi\}_{pi}\{\chi\}_{pi}^t[G]}{(2\lambda_iT_i + S_i)}[S(t)] \times \frac{[G]^t\{\chi\}_{pj}\{\phi\}_{pj}^t\{\beta\}}{(2\lambda_jT_j + S_j)}, \qquad i, j = 1, \ldots, 2l \tag{12}$$

and where $[S(t)]$ is a diagonal matrix with components $S_{ii}(t) = s_i^2(t)$. Note that for simple modulating functions $s_i(t)$ such as step functions, boxcar type functions, and exponential functions, a closed-form solution can be obtained for the modal cross-covariance quantities. The second-order statistics $\sigma_{\dot{r}}^2$ and $\sigma_{r\dot{r}}$ can be computed in a similar manner. More general stochastic excitations such as processes modeled as the output of a linear system (filter) with a Gaussian white noise input can also be treated in this formulation. In that case, an augmented system consisting of the original system and the input filter subjected to a white noise process has to be considered.

Design Points

The evaluation of the unconditional quantities $C(\{x\})$ and $P_{F_i}(\{x\}), i = 1, \ldots, K$, involves the computation of design points. The design points are found as the solution of the sublevel unconstrained optimization problem

$$\max_{\{b\}} J(\{x\} \mid \{b\}) P(\{b\}), \qquad \{b\} \in \Omega \tag{13}$$

where $J(\{x\} \mid \{b\})$ is a conditional quantity such as the conditional total cost and the conditional failure probabilities involved in the optimization problem (1). The main contribution to the unconditional quantity $J(\{x\})$ comes from the neighborhood of the design points. When multiple design points exist, it is necessary to find all local maxima. If one design point is neglected, its contribution to the value of the unconditional quantity may be significant, and, therefore, the estimate can be inaccurate.[15] In this formulation, a stochastic multistart technique is used to search for multiple design points.[18] The algorithm generates random initial points for the optimization process. Each starting point produces a local maximum that is computed by a standard local optimization scheme. Different stopping rules can be considered for the multistart technique. For example, the procedure stops when the number of different maxima that have been detected is equal to a user estimate of the total number of maxima. Another possibility is to estimate the relative size of the region of attraction in the set of uncertain system parameters Ω that the algorithm has not detected. In this context, a region of attraction is a subset of Ω such that if the optimization process starts at any point in the set, the optimum solution is unique and lies in Ω. The search of new local maxima is stopped as soon as the posterior expected relative size of the detected region exceeds a user specified number.[19]

In general, the search of design points occupies a considerable portion of the total computational effort during the optimization process. To evaluate the conditional quantities involved in the optimization problem (13), it is necessary to obtain the second-order statistics of system response functions. It is clear from Eq. (12) that the second-order statistics of the system responses depend on the modal energies T_i, S_i, and U_i, $i = 1, \ldots, 2l$, and the position parts of the right and left eigenvectors $\{\phi\}_i$ and $\{\chi\}_i$, $i = 1, \ldots, 2l$, respectively. At the same time, these quantities are implicit nonlinear functions of the vector of uncertain system parameters $\{b\}$. Therefore, the search of the design points implies the repeated evaluation of the system responses (structural analyses). For real systems, the evaluation of structural responses can be prohibitively expensive from the numerical point of view. To avoid this computational burden, approximation concepts are used for the evaluation of the second-order statistics.

Numerical Implementation

The fundamental ideas used in the approximation concepts method[12,13,20] are extended for the efficient evaluation of the second-order statistics of the system responses. In this approach, the complex modal energies T_i, S_i, and U_i, $i = 1, \ldots, 2l$, are chosen as intermediate response quantities, and they are approximated by using a convex linearization[21] with respect to the uncertain system parameters $\{b\}$. For example, modal energy T_i is approximated as

$$\tilde{T}_i = T_{i0} + \sum_{(+)} \frac{\partial T_i(\{b_0\})}{\partial b_j}(b_j - b_{j0}) + \sum_{(-)} \frac{\partial T_i(\{b_0\})}{\partial b_j}\frac{b_{j0}}{b_j}(b_j - b_{j0}) \tag{14}$$

where $T_{i0} = T_i(\{b_0\})$, $\{b_0\}$ is a point in Ω, $\sum_{(+)}$ means summation over the variables for which

$$\frac{\partial(\cdot)(\{b_0\})}{\partial b_j}$$

is positive, and $\sum_{(-)}$ contains the remaining variables. An attractive property of this linearization is that it yields the most conservative approximation among all of the possible combination of direct/reciprocal variables.[21] A similar approximation is used for the modal energies S_i and U_i. The partial derivatives used in the approximations are evaluated assuming that the position parts of the eigenvectors are invariant in the neighborhood of $\{b_0\}$. For example, the partial derivative of the modal energy T_i at $\{b_0\}$, with respect to b_j, is computed as

$$\left.\frac{\partial T_i}{\partial b_j}\right|_{\{b_0\}} = \{\chi\}_{pi}^t \left.\frac{\partial [M]}{\partial b_j}\right|_{\{b_0\}} \{\phi\}_{pi}$$

The same assumption is used for the evaluation of the partial derivatives $\partial U_i/\partial b_j$ and $\partial S_i/\partial b_j$, which are required for the approximations of the modal energies S_i and U_i. This assumption makes sensitivity calculation (derivatives) very inexpensive from a computational point of view. When the earlier approximations in Eq. (12) are introduced, an explicit approximation for the modal cross covariances $\Gamma_{ij}(t)$, $i, j = 1, \ldots 2l$, in terms of $\{b\}$ can be obtained. These approximations are then used to construct approximations for the second-order statistics of the system response $r_i(t, \{x\} \mid \{b\})$, that is, $\tilde{\sigma}^2_{r_i}$, $\tilde{\sigma}^2_{\dot{r}_i}$, and $\tilde{\sigma}_{r_i\dot{r}_i}$. Finally, the approximate second-order statistics of the response $r_i(t, \{x\} \mid \{b\})$ are used in combination with Eqs. (7) and (8) to estimate the conditional quantities. With these approximation concepts used, the optimization problem (13) can be replaced by the solution of a sequence of explicit approximate suboptimization problems.[20,21] At each stage of the iterative optimization process, the approximate subproblem, corresponding to problem (13), is constructed in terms of approximate modal energies. These quantities are approximated about the current design point $\{b_0\}$ as in Eq. (14). Note that only one exact structural analysis is required at each suboptimization problem. The invariant assumption of the mode shapes limits the relative change (move limits) in the optimization variables of the suboptimization problems, where the approximations are expected to yield reasonable results. However, numerical validations have shown that move limits up to 50% can be used without significant loss of accuracy in the final results.[17,20] Of course, the variability of the mode shapes can be considered explicitly in the approximations to increase the size of the move limits. A much faster convergence of the sequence of the explicit suboptimization problems is obtained in this case, but at the expense of more computational effort in the overall design process. Thus, the invariant assumption of the eigenvectors in the neighborhood of the current design point, where the approximations are constructed, is widely used in the context of structural optimization problems.

The explicit approximate problems as well as the top-level optimization problem, which corresponds to the overall optimization in the design variables $\{x\}$, are nonlinear optimization problems. In general nonlinear optimization problems, it is widely acknowledged that optimization algorithms using first-order information should be considered the most efficient algorithms. As a consequence of this, a first-order scheme is used in this formulation to solve the top-level as well as the approximate sublevel optimization problems.

Example Problem

The example problem consists of a generic primary–secondary system shown in Fig. 1. This type of system is chosen as a test problem because of the richness in its dynamic characteristics and the wide range of applications that this model has in engineering

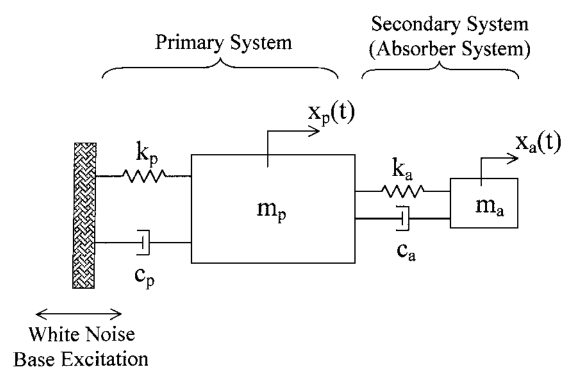

Fig. 1 Primary–secondary system.

vibration. Also, the behavior of this simple structure is representative of the behavior of more general multi-degree-of-freedom systems. In this application, the primary system represents a general structure, modeled as a single-degree-of-freedom system, whereas the secondary system serves as an absorber system consisting of a mass–damper–spring combination added to the primary system to protect it from vibrating. Absorbers are often used in a number of systems, including aircraft engines, building structures, transmission lines, and general rotating systems.[22] The parameters describing the structural system are specified as follows: mass of the primary system m_p, stiffness property of the primary system k_p, structural damping ratio of the primary system ξ_p ($\xi_p = c_p/2\sqrt{(k_p m_p)}$), mass of the absorber m_a, stiffness of the connection between the absorber and the primary system k_a, and damping ratio of the absorber ξ_a ($\xi_a = c_a/2\sqrt{(k_a m_a)}$). The masses are assumed to be fixed, and the mass ratio is taken to be $\mu = m_a/m_p = 0.01$. The combined system is subjected to a base acceleration that is modeled as a white noise process of duration equal to 10 times the natural period of the primary system in the absence of the absorber, that is, when $\mu = 0$. The system governing the evolution of the response has two complex conjugate modes. On the other hand, the equation for the evolution of the modal cross covariances has two real components and four components that appear in complex conjugate pairs.

The objective of the example problem is to find the optimal design of the absorber that minimizes the total cost, including the initial construction cost and the expected cost of failure. At the same time, the system is subjected to a reliability constraint that is given in terms of the displacement response of the primary system relative to its base. The design variables are the absorber parameters k_a and ξ_a. The uncertain system parameters are chosen to be the stiffness property of the primary system k_p and the damping ratio of the primary system ξ_p, with most probable values $\hat{k}_p$ and $\hat{\xi}_p$, respectively. The uncertain parameters are parameterized and written as $k_p = \hat{k}_p b_1$ and $\xi_p = \hat{\xi}_p b_2$, where b_1 and b_2 are independent and lognormally distributed with the most probable value equal to 1.0 and standard deviation σ_1 and σ_2, respectively. The most probable value of the period of the primary system is chosen to be 0.4 s, and $\hat{\xi}_p = 0.01$. For illustration purposes, the initial construction cost is defined as a linear function of the damping ratio of the absorber and given by $C_c(\xi_a) = 10^5 \times 5.05\xi_a + 505.00$. The cost of failure C_F, for known primary system parameters k_p and ξ_p, is the product between a cost, taken as 10^5, and the probability of failure of the absorber. Failure is assumed to occur when the restoring force of the spring connecting the absorber to the primary system reaches some critical level for the first time. Thus, the response quantity of interest in this case is $r(t) = k_a[x_a(t) - x_p(t)]$, where $x_a(t)$ is the displacement of the absorber and $x_p(t)$ the displacement of the primary system. The threshold level value is assumed to be four times the standard deviation of the restoring force response of the initial design. The reliability constraint is given in terms of the failure probability of the primary system P_F. In this case, failure is assumed to occur when the displacement of the primary system relative to the base exceeds some critical level for the first time. The threshold level is chosen to be four times the standard deviation of the relative displacement response of the primary system in the absence of the absorber. The specified level of failure is taken to be equal to $P_F^* = 2 \times 10^{-2}$.

To gain insight into the effect of system uncertainties on the objective function and reliability constraint, the cost of failure C_F and the failure probability P_F are evaluated as functions of the uncertain system parameters b_1 and b_2. These functions are shown in Figs. 2 and 3, respectively, and they are evaluated at the initial design $k_{a(\text{initial})}$ and $\xi_{a(\text{initial})}$. The following values for the initial design are assumed: $k_{a(\text{initial})}/\hat{k}_p = 6.4 \times 10^{-3}$, where $\hat{k}_p$ is, as before, the most probable value of the stiffness property of the primary system, and $\xi_{a(\text{initial})} = 1\%$. It is found that the failure probability at the most probable values of the uncertain parameters ($b_1 = b_2 = 1.0$) is smaller than P_F^* and equal to $P_F = 2.95 \times 10^{-3}$. Therefore, the initial design is feasible in this situation. However, it will be shown that the feasibility of the initial design can be altered because of the effect of the uncertainties in the system parameters. It is also observed from Figs. 2 and 3 that, when the stiffness of the primary system is about 50% of its most probable value ($b_1 = 0.5$), that is,

Table 1 Expected failure probability estimates (reliability constraint), initial design

Case	Exact	Perturbation	Asymptotic
1	0.29×10^{-2}	——	——
2	1.69×10^{-2}	1.57×10^{-2}	1.60×10^{-2}
3	5.76×10^{-2}	2.58×10^{-2}	5.69×10^{-2}

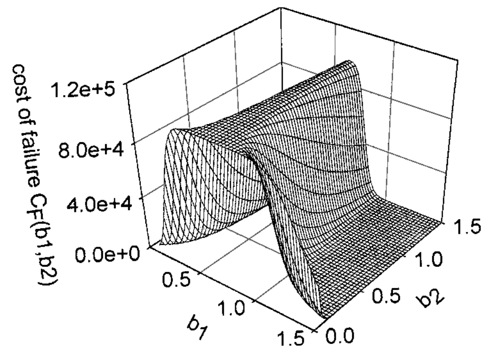

Fig. 2 Cost of failure at the initial design as a function of the uncertain system parameters b_1 and b_2.

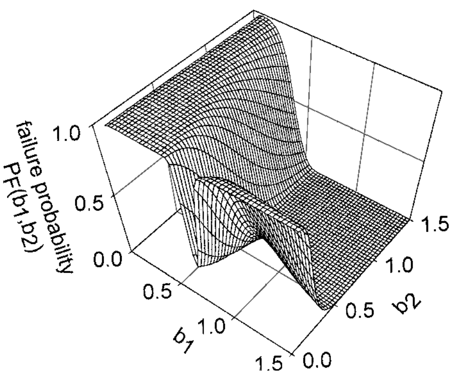

Fig. 3 Failure probability of the primary system at the initial design as a function of the uncertain system parameters b_1 and b_2.

$k_p = 0.5\hat{k}_p$, the failure probability is very small. In this situation, the motion of the primary system is taken by the motion of the absorber, and, therefore, this device offers an effective protection to the primary system by reducing its vibration magnitude. This is especially true if the structural damping ratio ξ_p is small. At the same time, it is seen that the cost of failure of the absorber is maximized under this condition. This is reasonable because the motion of the vibration absorber is large in this situation. Of course, the preceding comments are only valid under the condition $k_p = 0.5\,\hat{k}_p$ because the uncertainty in the system parameters can very easily destroy the effectiveness of the absorber operation.

Table 1 shows the value of the expected failure probability (unconditional) at the initial design obtained by the importance sampling technique, second-order perturbation method, and asymptotic approximation. For discussion purposes, the value obtained by the importance sampling technique with a large number of samples (2000) is taken as the exact value. The following cases, namely, 1, 2, and 3, are considered. In case 1 the stiffness property and the damping ratio of the primary system are taken at their most probable values, that is, $k_p = \hat{k}_p$ and $\xi_p = \hat{\xi}_p$. In the other two cases, the level of uncertainty of ξ_p is fixed at $\sigma_2 = 0.25$ (standard deviation), and two levels of uncertainty of k_p, namely, $\sigma_1 = 0.25$ (case 2), and $\sigma_2 = 0.40$ (case 3), are considered. These levels of uncertainty represent a variability of the natural frequency of the primary system of approximately 10 and 20% for cases 2 and 3, respectively. The asymptotic method performs very well in terms of predicting the value of the expected failure probability of the primary system for cases 2 and 3. Contrarily, the second-order perturbation method gives poor results. In fact, for high levels of uncertainty, it underestimates the results by a factor of more than two. This result is expected because the second-order perturbation method is based on the expansion of the failure probability function $P_F(b_1, b_2)$ into a second-order Taylor series about the most probable values of b_1 and b_2. The local expansion is not able to capture the nonlinearity of the failure probability function in the space of uncertain system parameters Ω (see Fig. 3). It is also clear that the failure probability estimate based on the most probable primary system (case 1) is highly underestimated. Thus, neglecting the uncertainties in the system parameters will give unreliable results for the failure reliability estimate. This, in turn, will produce an important impact on the optimal design.

It is found that for cases 2 and 3 there exist two design points for the failure probability function $P_F(\{b\})P(\{b\})$ and one design point for the failure cost function $C_F(\{b\})P(\{b\})$. Figure 4 shows the integrand $P_F(\{b\})P(\{b\})$ for case 3 at the initial design. In this case, the contribution of the first design point to the failure probability is 14%, whereas the importance of the second design point is 86%. The relative contribution reflects the importance of the design points in the reliability computation, and it is controlled by the current design in the optimization process. To illustrate this point, Fig. 5 shows the integrand function $P_F(\{b\})P(\{b\})$ at the optimal design. The importance of the first design point increases to 37%, whereas the second design point decreases to 63%. Figures 6 and 7 show the integrand function $C_F(\{b\})P(\{b\})$ for case 3 at the initial and final design, respectively. The asymptotic estimate of the expected failure cost at the initial design is equal to 0.34×10^5. This value is then reduced more than 40 times at the final design. This reduction shows the effectiveness of the optimization process in reducing the initial expected failure cost of the design. A global optimization strategy is used in this implementation to calculate the design points [problem (13)]. The algorithm uses a stochastic multistart technique together with version 4.0 of DOT,[23] which is a first-order numerical optimizer. For all initial sample points generated by the multistart technique, convergence to a local maximum is obtained in fewer than 15 cycles, when approximation concepts are used. On the other hand, the number of analyses required for convergence is more than 200 if the sublevel optimization problem (13) is solved using exact system analyses. Clearly, the use of approximation concepts allows a considerable reduction in the number of analyses required to obtain the design points. Reduction in computational time is even more substantial for the design of complex systems because the computational burden of each structural analysis is very large. Numerical results have shown that the computational requirements can by reduced by a factor of more than 20 when approximation concepts are used. Thus, the feasibility of the proposed approach to complex system is apparent.

Tables 2 and 3 show the final designs for cases 1, 2, and 3 obtained by using exact and asymptotic estimates for the unconditional quantities involved in the optimization problem, respectively.

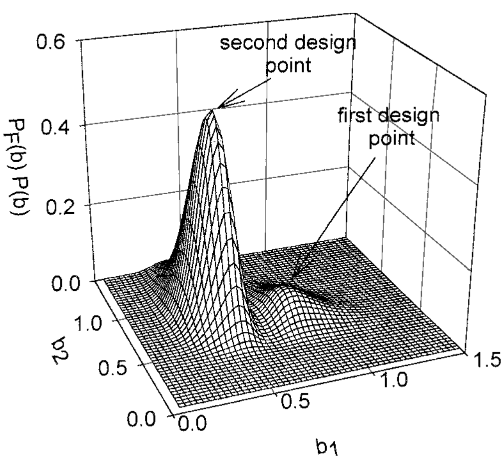

Fig. 4 Failure probability function of the primary system at the initial design as a function of the uncertain system parameters b_1 and b_2.

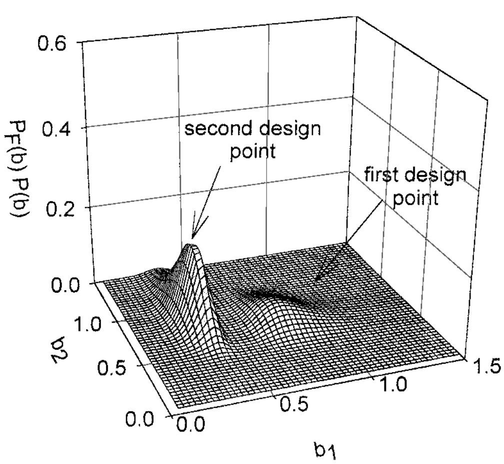

Fig. 5 Failure probability function of the primary system at the final design as a function of the uncertain system parameters b_1 and b_2.

Table 2 Final designs (exact)

Design variable	Case 1	Case 2	Case 3
$k_a/k_{a(\text{initial})}$	0.10	0.83	0.79
$\xi_a/\xi_{a(\text{initial})}$	0.10	2.08	5.47
Total cost	0.110×10^{-2}	0.136×10^5	0.279×10^5

Table 3 Final designs (asymptotic approximations)

Design variable	Case 1	Case 2	Case 3
$k_a/k_{a(\text{initial})}$	0.10	0.83	0.79
$\xi_a/\xi_{a(\text{initial})}$	0.10	2.07	5.10
Total cost	0.110×10^{-2}	0.133×10^5	0.264×10^5

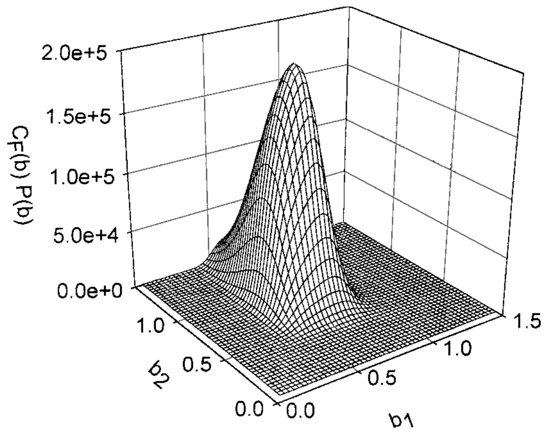

Fig. 6 Failure cost function at the initial design as a function of the uncertain system parameters b_1 and b_2.

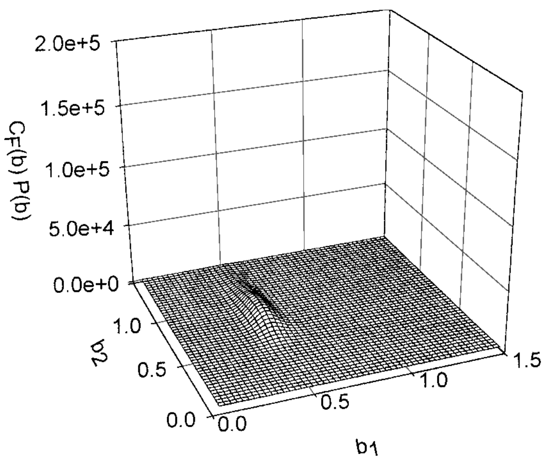

Fig. 7 Failure cost function at the final design as a function of the uncertain system parameters b_1 and b_2.

The side constraints for the design variables are set as follows: $0.1 \le k_a/k_{a(\text{initial})} \le 4.0$ and $0.1 \le \xi_a/\xi_{a(\text{initial})} \le 10.0$, where $k_{a(\text{initial})}$ and $\xi_{a(\text{initial})}$ are, as before, the values of the design variables at the initial design. Note that the optimal design corresponding to case 1, that is, when the most probable values of the primary system parameters are considered, is very different from those of cases 2 and 3. In the conditional case (case 1), the entire design space is feasible, and the minimum cost is obtained at the lower bound values of the design variables. The failure probability at the optimal design P_F is such that $P_F/P_F^* = 0.33$, and, therefore, the optimal solution is not active in this case. This result is illustrated in Fig. 8, where the constraint function (expected failure probability of the primary system) is shown as a function of the design variables. When uncertainties are considered, the optimal solution is active and the design variables lie inside the design space as shown in Fig. 9. In Fig. 9, the objective function (initial construction cost plus expected cost of failure of the absorber system) is shown as a function of the design variables for case 3. From Fig. 9 and Tables 2 and 3, it is clear that the uncertainties are important in the optimization process because they can change the optimal design dramatically. In fact, the total cost of the optimal design in the conditional case is 10^7 times less than that of the optimal design obtained when uncertainties in the primary system parameters are considered. This result indicates that the optimal solution can be highly sensitive to variation in the system parameters. Note that the shape of the integrands of the failure probability integrals changes during the optimization process. The numerical results shown in Tables 2 and 3 indicate that the asymptotic approximation is able to capture the quantitative behavior of the probability integrals during the entire optimization process for the level of uncertainties considered in this study. For higher levels of uncertainty (standard deviation more than 50%), the accuracy of the asymptotic approximations deteriorates,[10] and the final design can be affected significantly. In this case, the accuracy of the estimates can be improved to any desired, by using, for example, importance sampling techniques. As mentioned, the main difference in this situation is in the number of structural analyses required during the optimization process.

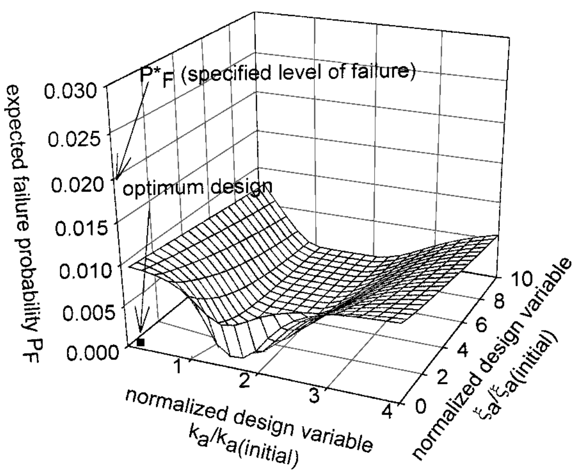

Fig. 8 Expected failure probability of the primary system as a function of the design variables (case 1).

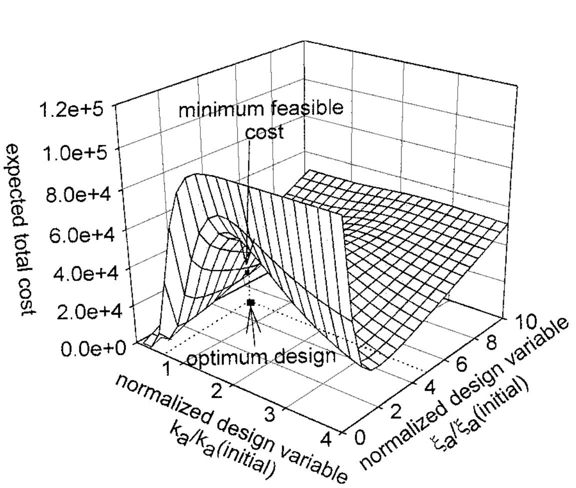

Fig. 9 Expected total cost as a function of the design variables (case 3).

Table 4 Constraint violations P_F/P_F^*

Case	Conditional optimal design	Unconditional optimal design
2	3.57	1.01
3	6.81	1.05

The importance of considering the effects of uncertainty in the system parameters during the optimization process can also be illustrated from a constraint violations point of view. Table 4 shows the constraint violations at the conditional optimal design (case 1), and unconditional optimal design obtained by the asymptotic approximation method. In Table 4, the exact value of the unconditional failure probability P_F at different optimal designs is computed by the importance sampling technique (exact value). For example, if the failure probability is evaluated at the conditional optimal design, a factor of more than six is obtained in case 3. Thus, the failure probability of the conditional optimal design is more than six times the value of the failure probability constraint P_F^*, when the uncertainty in the primary system parameters is considered. It is observed that the constraint violations at the optimal solution obtained by the proposed implementation are small, even for the case with large uncertainties (case 3). Finally, for completeness the behavior of the proposed optimal dynamic absorber is compared with absorbers obtained by conventional deterministic approaches.[22] For example, the optimal tuned mass–damper corresponding to case 3 of the example problem has a probability of failure equal to 6.75×10^{-2} and a final cost equal to 0.508×10^5, when the uncertainty in the primary system parameters is considered. It is clear that this design is infeasible and much more expensive than the design obtained by the proposed formulation. Therefore, it is recommended that the procedure described in this paper be used instead of deterministic tuning approaches whenever there are uncertainties in the primary system properties. These results also indicate, once again, the importance of considering the effect of uncertainty explicitly during the design process.

Conclusions

A general solution strategy for the reliability-based optimization of uncertain linear structural systems subjected to stochastic excitation has been presented. An asymptotic approximation method is used to derive estimates for unconditional reliabilities in a numerically efficient way. The combination of the asymptotic method with approximation concepts allow a considerable reduction in the number of exact system analysis required to obtain reliability estimates. Numerical results have shown that uncertainty in the model parameters may cause significant changes in the optimal design of systems subjected to stochastic loads. In these situations, the errors or uncertainties in the specification of the system properties should be properly accounted for during the optimization process. Also, the accuracy of reliability estimates plays an important role in the optimization problem. Inaccurate reliability estimates may produce infeasible optimal designs, and, therefore, the optimization process may lead to misleading conclusions regarding the feasibility and safety of the optimized system. Finally, as the simple yet illustrative example demonstrates, the proposed methodology provides a general framework in which the optimal design of structures with uncertain properties subjected to stochastic excitations can be determined. The implementation of the methodology to more complex structural systems is immediate. Thus, the proposed implementation is expected to be useful in the optimal design of real structural systems.

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